CHAPTER 20

Centre of Gravity

20.1. INTRODUCTION

Definition : *The centre of gravity of a body is the point at which the resultant of the weights of all the panicles of the body acts, whatever may be the orientation of the body. The total weight of the body may be supposed to act at its centre of gravity.*

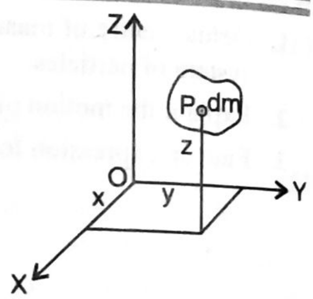

Fig. 20.1

Suppose the particles *A, B, C,* .... of a body have masses m_1, m_2, m_3, ... Let their coordinates in a rectangular cartesian coordinate system be (x_1, y_1, z_1), (x_2, y_2, z_2).... (x_n, y_n, z_n).

Then, the coordinates of the centre of gravity *G* of the body are

$$\bar{x} = \frac{\sum m_n x_n}{\sum m_n};\ \bar{y} = \frac{\sum m_n y_n}{\sum m_n};\ \bar{z} = \frac{\sum m_n z_n}{\sum m_n};$$

Suppose an element *P* of the body has a mass *dm* (Fig. 20.1) and its coordinates are *x, y, z.* Then,

$$\bar{x} = \frac{\int x\,dm}{\int dm} = \frac{1}{M}\int x\,dm,\ \bar{y} = \frac{1}{M}\int y\,dm,\ \bar{z} = \frac{1}{M}\int z\,dm$$

Here, the integrals extend over all elements of the body, and

$M = \int dm$ = Total mass of the body.

20.2. CENTRE OF GRAVITY OF A RIGHT SOLID CONE

Let *ABC* represent a solid cone of height *h* and semi-vertical angle α (Fig. 20.2). The cone may be considered to be made up of a large number of circular discs parallel to the base. The centre of gravity of each disc lies at its centre. Therefore, the C. G., of the cone should lie along the axis *AD* of the cone.

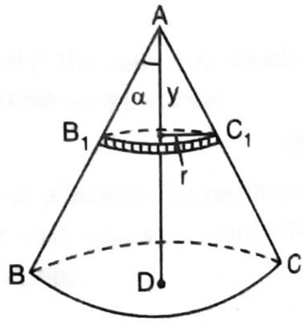

Fig. 20.2

Consider a disc B_1C_1 of thickness *dy* at a distance *y* below the vertex *A*. If *r* is the radius of the disc, then

$$r = y\tan\alpha$$

Volume of the disc = Area × thickness = $\pi y^2 \tan^2\alpha\,dy$

Mass of the disc = $dm = \pi y^2 \rho \tan^2\alpha\,dy$.

where ρ = density of the cone.

The distance of the *C. G.* of the cone from the vertex is given by

$$\bar{y} = \frac{\int y\,dm}{\int dm} = \frac{\int_0^h \pi y^3 \rho \tan^2\alpha\,dy}{\int_0^h \pi y^2 \rho \tan^2\alpha\,dy} = \frac{\int_0^h y^3\,dy}{\int_0^h y^2\,dy} = \frac{3}{4}h.$$

Therefore, the $C.\ G.$, of the cone is along its axis at a distance of $\frac{3}{4}h$ from the vertex.

20.3. CENTRE OF GRAVITY OF A SOLID HEMISPHERE

Let ABC represent a solid hemisphere of radius r, centre O and density ρ (Fig. 20.3). Consider an elementary slice of the hemisphere with radius y and thickness dx, at a distance x from O.

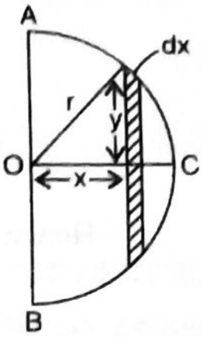

Fig. 20.3

Volume of the slice $= \pi y^2\, dx = \pi(r^2 - x^2)\, dx$.

Mass of the slice $= dm = \rho\pi\,(r^2 - x^2)\, dx$.

The distance of the $C.\ G.$, of the hemisphere from O is given by

$$\bar{x} = \frac{\int x\, dm}{\int dm} = \frac{\int_0^r x\,\rho\pi\,(r^2 - x^2)dx}{\int_0^r \rho\pi\,(r^2 - x^2)dx} = \frac{\int_0^r (r^2x - x^3)dx}{\int_0^r (r^2 - x^2)dx}$$

$$\therefore \quad \bar{x} = \frac{3}{8}r.$$

Hence, the $C.\ G.$, of the solid hemisphere is on its axis at a distance $\frac{3}{8}r$ from the centre.

20.4. CENTRE OF GRAVITY OF A HOLLOW HEMISPHERE

Let ACB be a section of a hemisphere of radius r, centre O and surface density ρ [Fig, 20.4]. Imagine the surface of the hemisphere to be divided into slices like PQQ_1P_1 by planes parallel to AB. If $\angle POC = \theta$ and $\angle POQ = d\theta$, then

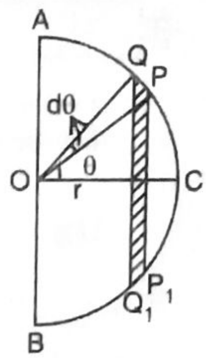

Fig. 20.4

radius of the ring $= r \sin\theta$

width of the ring $= r\, d\theta$

Area of the ring $= 2\pi r \sin\theta.\ r d\theta$

$\therefore$ mass of the ring $= dm = 2\pi r^2\, \rho \sin\theta\, d\theta$.

The $C.\ G.$, of this ring is at the centre of the ring at a distance $r\cos\theta$ from O.

The distance of the $C.\ G.$, of the hollow hemisphere from O is given by

$$\bar{x} = \frac{\int x\, dm}{\int dm} = \frac{\int_0^{\pi/2} (r\cos\theta)\, 2\pi r^2 \rho \sin\theta\, d\theta}{\int_0^{\pi/2} 2\pi r^2 \rho \sin\theta\, d\theta} = \frac{\int_0^{\pi/2} r \sin\theta \cos\theta\, d\theta}{\int_0^{\pi/2} \sin\theta\, d\theta}$$

$$\therefore \quad \bar{x} = r/2$$

The $C.$ G., of a hollow hemisphere is on its axis at a distance $r/2$ from the centre, *i.e.*, the centre of gravity is at the mid point of the radius OC.

20.5. CENTRE OF GRAVITY OF A SOLID TETRAHEDRON

Let $ABCD$ be the tetrahedron and G_1, the centre of gravity of the base BCD (Fig. 20.5). Let h be the altitude of the tetrahedron and ρ its density. Suppose the tetrahedron is divided into thin slices by planes parallel to the base BCD. Consider one such slice $B_1C_1D_1$ of thickness dx at a depth x below A. Let S be the area of the triangular base BCD. Then we have, $\frac{B_1C_1}{BC} = \frac{x}{h}$.

If a_1 and a are the altitudes of triangles $B_1C_1D_1$ and BCD respectively,

$$\frac{a_1}{a} = \frac{x}{h}.$$

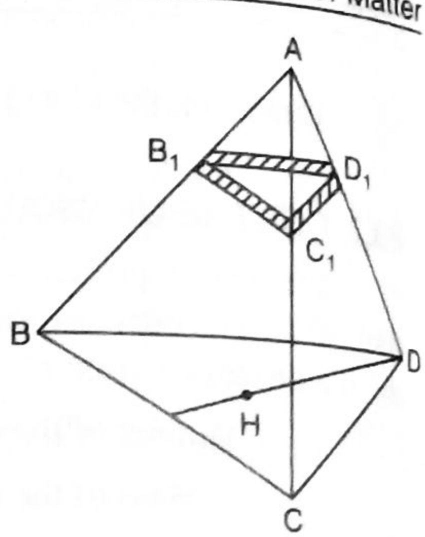

Fig. 20.5

Now, area of $\Delta B_1C_1D_1 = \frac{1}{2} B_1C_1 \times a_1$

Area of $\Delta BCD = \frac{1}{2} BC \times a = S.$

Hence, $\frac{\text{Area of } \Delta B_1C_1D_1}{S} = \frac{B_1C_1}{BC} \times \frac{a_1}{a} = \frac{x^2}{h^2}$

$\therefore$ Area of $\Delta B_1C_1D_1 = Sx^2/h^2$

Volume of the slice $B_1C_1D_1 = Sx^2dx/h^2$

Mass of the slice $= dm = \rho Sx^2dx/h^2$

The distance of the centre of gravity of the tetrahedron from A is given by

$$\bar{x} = \frac{\int x\, dm}{\int dm} = \frac{\int_0^h x\rho Sx^2 dx/h^2}{\int_0^h \rho Sx^2 dx/h^2} = \frac{\int_0^h x^3 dx}{\int_0^h x^2 dx} = \frac{3}{4}h$$

Hence, the $C.G.$, of a uniform tetrahedron lies at a point G on the line AH such that $AG : GH = 3:1$.

EXERCISE XX

1. Find the position of C.G., in the following cases: (*i*) Right solid cone, (*ii*) solid hemisphere, (*iii*) hollow hemisphere and (*iv*) solid tetrahedron.

22 CHAPTER

Friction

22.1. INTRODUCTION

Suppose we push a book along a table, giving it some velocity. After we release it, it slows down and finally stops. This indicates that a force is opposing the motion. Whenever one surface moves in contact with another, a force is generated that tends to retard the motion. This force is called friction.

Friction plays a vital role in our life. For example, man is able to walk on the road because of friction between his feet and the ground. In machines friction reduces efficiency. Lubrication and ball bearings reduce friction in machines.

22.2. STATIC, DYNAMIC, ROLLING AND LIMITING FRICTION

The frictional forces acting between surfaces at rest with respect to each other are called forces of *static friction*. The maximum value of the frictional force between two bodies in contact is called *limiting friction*. The forces acting between surfaces in relative motion are called *kinetic* (or *dynamic) friction*. The frictional force between two surfaces when one rolls over the other is called *rolling friction*. It is easy to roll a cylindrical body than to slide it. Thus, the rolling friction is less than the sliding friction. We require more force to start the motion of a body than to keep it in uniform motion. Thus, static friction is larger than dynamic friction.

22.3. LAWS OF STATIC FRICTION

1. The direction of the frictional force is always opposite to the direction in which one body tends to slide over another.

2. The magnitude of the force of friction when there is equilibrium between two bodies is just sufficient to prevent the motion of one body over the other. The frictional force attains a maximum value when one body is just on the point of sliding over the other. The maximum value of the force of friction is called limiting friction.

3. The magnitude of the force of limiting friction bears a constant ratio to the normal reaction between the two bodies. This ratio is called *coefficient of friction* and is denoted by μ. If F is the limiting friction and R the normal reaction between the two bodies, then $\mu = F/R$ · μ depends only on the nature of surfaces in contact.

4. The limiting friction is independent of the extent and shape of the surfaces in contact, provided the normal reaction is unaltered.

5. When a body is in motion, the direction of friction is still opposite to the direction of motion of the body and is independent of the velocity. But the ratio of the force of friction to the normal reaction is slightly less than that when the body is just on the point of motion.

Angle of friction: Let F be the force of limiting friction and R, the normal reaction. Let S be the resultant of these two forces (Fig. 22.1). Then the angle which this resultant force makes with the normal reaction is called the *angle of friction*. It is denoted by λ.

$$\text{Then, } \tan\lambda = \frac{F}{R} = \frac{\mu R}{R} = \mu \left(\because \mu = \frac{F}{R}\right)$$

Cone of Friction: Consider a cone with the point of contact of two bodies as the vertex, the normal reaction as axis and semi-vertical angle λ. Then the resultant reaction (S) may lie anywhere within or on the surface of the cone (Fig.22.1). This imaginary cone is called the *cone of friction*.

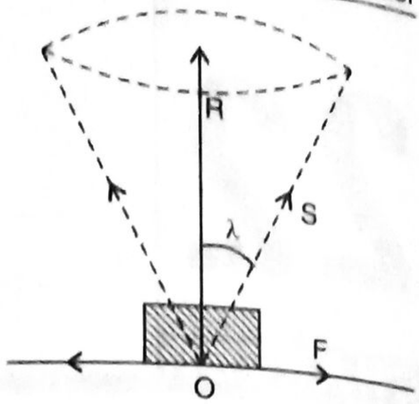

Fig. 22.1

22.4. EXPERIMENTAL METHOD FOR DETERMINING COEFFICIENT OF FRICTION BETWEEN TWO SURFACES

Suppose a body of weight *mg* is placed on a rough inclined plane (Fig. 22.2). The inclination is increased till a position is attained when the body just slides. At this position let the inclination of the plane to the horizontal be α. The forces acting on the body are: (*i*) the weight *mg* acting vertically downwards, (*ii*) the normal reaction *R* perpendicular to the plane and (*iii*) the force of limiting friction μ*R* acting up the plane. Resolve *mg* into a component *mg* sin α down the plane and a component *mg* cos α at right angles to the plane. When the equilibrium is limiting,

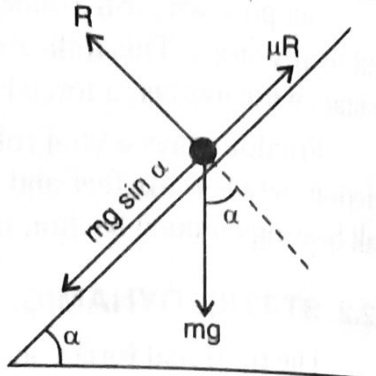

Fig. 22.2

$$mg \sin\alpha = \mu R \quad \text{...}(i)$$

$$mg \cos\alpha = R \quad \text{...}(ii)$$

Dividing (*i*) by (*ii*), $\tan\alpha = \mu$

22.5. EQUILIBRIUM OF A BODY ON A ROUGH INCLINED PLANE ACTED UPON BY AN EXTERNAL FORCE

Proposition: *A body of weight w is in equilibrium on a rough inclined plane of angle α > λ under the action of an external force inclined upwards at an angle θ with the plane. Find the value of P for limiting equilibrium.*

Case 1: Let the body be just on the point of sliding down the plane. Let *P* be the magnitude of the external force, applied at an angle θ with the plane. The forces acting on the body are: (*i*) The weight of the body (*w*) acting vertically down, (*ii*) The normal reaction (*R*) acting perpendicular to the plane (*iii*) The force of limiting friction (μ*R*) acting up the plane (*iv*) The external force (effort) *P* making an angle θ with the line of gratest slope of the inclined plane (Fig. 22.3). Resolving the forces parallel and perpendicular to the plane.

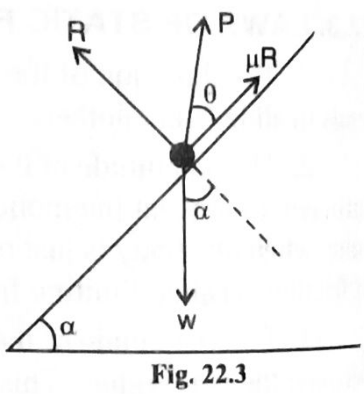

Fig. 22.3

$$P\cos\theta + \mu R = w\sin\alpha \quad \text{...}(1)$$

$$P\sin\theta + R = w\cos\alpha \quad \text{...}(2)$$

Multiplying (2) by μ and subracting from (1),

$$P(\cos\theta - \mu\sin\theta) = w(\sin\alpha - \mu\cos\alpha)$$

or

$$P = w\frac{(\sin\alpha - \mu\cos\alpha)}{(\cos\theta - \mu\sin\theta)}$$

But $\mu = \tan\lambda$, where λ is the angle of friction.

Hence $$P = w\frac{(\sin\alpha - \tan\lambda\cos\alpha)}{(\cos\theta - \tan\lambda\sin\theta)} = w\frac{(\sin\alpha\cos\lambda - \sin\lambda\cos\alpha)}{(\cos\theta\cos\lambda - \sin\lambda\sin\theta)}$$

$$\therefore \quad P = \frac{w\sin(\alpha - \lambda)}{\cos(\theta + \lambda)} \quad ...(3)$$

Case 2: Let the body be just on the point of sliding up the plane. Let P_1 be the magnitude of the external force. In this case, the force of limiting friction (μR) acts down the plane.

Resolving the forces parallel and perpendicular to the plane,

$$P_1 \cos\theta = w\sin\alpha + \mu R \quad ...(4)$$

$$P_1 \sin\theta + R = w\cos\alpha \quad ...(5)$$

Simplifying, we get, $$P_1 = \frac{w\sin(\alpha + \lambda)}{\cos(\theta - \lambda)} \quad ...(6)$$

Corollary 1: P_1 is a minimum when $\cos(\theta - \lambda)$ is maximum *i.e.*, when $\cos(\theta - \lambda) = 1$ i.e., when $\theta = \lambda$. Hence force required to move the body up the plane will be least when it is applied in a direction making with the inclined plane an angle equal to the angle of friction.

Corollary 2: Let a body be at rest on a rough inclined plane whose inclination to the horizontal $\alpha > \lambda$. Let it be acted upon by an external force applied parallel to the plane. Here $\theta = 0$. From (3) and (6),

$$P = \frac{w\sin(\alpha - \lambda)}{\cos\lambda} \quad ...(7)$$

and $$P_1 = \frac{w\sin(\alpha + \lambda)}{\cos\lambda} \quad ...(8)$$

EXERCISE XXII

1. Define friction. What is the cone of friction ?
2. Define limiting friction, coefficient of friction, angle of friction and cone of friction. Show that coefficient of friction is related to angle of friction by $\mu = \tan\lambda$.
3. State the laws of friction. (*Madras 1995, 1992*)
4. A block is placed on an inclined plane of inclination θ. Show that to keep the block at rest, the external force P must lie between $\frac{w\sin(\theta \pm \lambda)}{\cos\lambda}$ where λ is angle of friction.